Quest for Mathematics I (E2): Exercise sheet 1

1. Find the limits as $n \to \infty$ of the following expressions.

   (a) $\left(\frac{3n}{n+3}\right)^2$

   (b) $\left(\sqrt{n+2} + \sqrt{n}\right)\left(\sqrt{n+1} - \sqrt{n}\right)$

   (c) $5^n - 3^n$

   (d) $(5^n - 3^n)^{1/n}$

   (e) $\frac{n!}{2^n}$

2. Find all the possible limits of
$$\frac{1 - r^n}{1 + r^n}$$
   as $n \to \infty$, where $r \in \mathbb{R}\backslash\{-1\}$.

3. The sequence $(a_n)_{n \geq 1}$ satisfies:
$$a_1 = 1, \qquad a_{n+1} = 2 + \frac{3}{4}a_n.$$

   Explain why the sequence converges, and find its limit.

4. The sequence $(F_n)_{n \geq 1}$ satisfies $F_1 = F_2 = 1$, and
$$F_{n+1} = F_n + F_{n-1}.$$

   Suppose it is known that
$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = f$$
   for some $f > 0$. Find f.